An Outer Approximation Method for Minimizing the Product of Several Convex Functions on a Convex Set*

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Abstract. This paper addresses the minimization of the product of p convex functions on a convex set. It is shown that this nonconvex problem can be converted to a concave minimization problem with p variables, whose objective function value is determined by solving a convex minimization problem. An outer approximation method is proposed for obtaining a global minimum of the resulting problem. Computational experiments indicate that this algorithm is reasonable efficient when p is less than 4.

Key words. Nonconvex minimization, global optimization, outer approximation method, convex multiplicative function.

1. Introduction

The purpose of this paper is to propose a practical algorithm for solving a nonconvex minimization problem:

$$\left|\begin{array}{ll} \text{minimize} & \prod_{j=1}^{p} f_j(x) \\ \text{subject to} & x \in X , \end{array}\right. \tag{1.1}$$

where $f_j: R^n \rightarrow R^1$, $j = 1, \ldots, p$ are nonnegative convex functions on a convex set $X \subset R^n$. It is well known [10] that the product of convex functions need not be convex and thus (1.1) belongs to a class of global optimization problems. At the same time, (1.1) has many practical applications in such areas as microeconomics [5], VLSI chip design [16], bond portfolio optimization [9], or multicriteria optimization problems [4] to name only a few.

For example, to solve a variant of multi-objective bond portfolio optimization model [9], we have to minimize the product of $2 \sim 4$ affine and/or linear fractional functions over a polytope defined by m linear equations and n non-negative variables where $m \leqslant 50$ and n is several hundreds. According to [8], a standard reference in multi-objective optimization problems, the number of objectives in a

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majority of real world problems is at most seven and usually less than four. Thus an efficient algorithm for solving (2.1) for p up to 4 is of primary importance in the field of multi-objective optimization.

In [10], the authors proposed a parametric simplex algorithm for obtaining a global minimum of a special class of (1.1), in which $p = 2$ and f_j's are affine. In a subsequent paper [14], we proposed a parametric successive underestimation method for $p = 2$ with nonnegative convex functions f_j's. As shown in [10, 14], the proposed algorithms provide us with very efficient methods for solving these special problems.

The remarkable success of "parameterization" techniques for linear and convex multiplicative programming problems motivated us to extend them to yet another class of global optimization problems. Readers are referred to [23, 12, 22] for the recent progress in these directions.

In this paper, we develop a practical algorithm for a more general and more difficult class of global optimization problems by extending the "parameterization" technique referred to above.

In Section 2, we will show that (1.1) can be converted to a concave minimization problem in a p-dimensional space by introducing p auxiliary variables (parameters). In Section 3, we will propose an outer approximation algorithm for obtaining a globally ϵ-optimal solution of (1.1) in finitely many steps, by exploiting the special structure of the p-dimensional problem. Results of computational experiments for p up to five are presented in Section 4, which demonstrates that our algorithm is practical for the problem (1.1) up to at least $p = 4$.

2. Definition of the Master Problem

Let us consider a nonconvex minimization problem defined below:

$$(\mathrm{P})\left|\begin{array}{ll} \text{minimize} & g_0(x) = \prod_{j=1}^{p} f_j(x) \\ \text{subject to} & g_i(x) \leq 0\,, \qquad i = 1, \ldots, m\,, \end{array}\right. \tag{2.1}$$

where f_j: $R^n \to R^1$, $j = 1, \ldots, p$ and g_i: $R^n \to R^1$, $i = 1, \ldots, m$ are convex functions. We assume in the sequel that the feasible region:

$$X = \{x \in R^n \mid g_i(x) \leq 0, \quad i = 1, \ldots, m\} \tag{2.2}$$

is nonempty and bounded and that

$$f_j(x) \geq 0\,, \quad \forall x \in X\,, \quad j = 1, \ldots, p\,. \tag{2.3}$$

If there exists some $f_j(x)$ which attains its lower bound zero at $x^j \in Z$, then x^j is obviously an optimal solution of (P). This can be checked by solving p convex minimization problems:

$$\text{minimize}\{f_j(x) \mid x \in X\}\,, \quad j = 1, \ldots, p\,. \tag{2.4}$$

Thus, we can rewrite the assumption (2.3) without loss of generality as follows:

$$f_j(x) > 0\,, \quad \forall x \in X\,, \quad j = 1, \ldots, p\,. \tag{2.5}$$

Let us introduce a vector of auxiliary variables $\xi = (\xi_1, \ldots, \xi_p)^t$ and define the following master problem:

$$\left|\begin{array}{ll} \text{minimize} & G(x, \xi) = \sum_{j=1}^{p} \xi_j f_j(x) \\ \text{subject to} & x \in X\,, \\ & \prod_{j=1}^{p} \xi_j \geq 1\,, \quad \xi \geq 0\,. \end{array}\right. \tag{2.6}$$

THEOREM 2.1. *Problem* (2.6) *has an optimal solution* (x^*, ξ^*) *such that* x^* *is an optimal solution of* (P). *Also, the following relations hold*:

$$G(x^*, \xi^*) = p\sqrt[p]{g_0(x^*)}\,, \tag{2.7}$$

$$\xi_j^* = \sqrt[p]{g_0(x^*)}/f_j(x^*)\,, \quad j = 1, \ldots, p\,. \tag{2.8}$$

Proof. By the assumption (2.5), the value of $\min\{\Sigma_{j=1}^{p} f_j(x)\xi_j \mid \Pi_{j=1}^{p} \xi_j \geq 1, \xi \geq 0\}$ is finite for any $x \in X$. Therefore, (2.6) must have an optimal solution (x^*, ξ^*) because X is assumed to be nonempty and bounded.

Let $\xi(x) = \operatorname{argmin}\{\Sigma_{j=1}^{p} f_j(x)\xi_j \mid \Pi_{j=1}^{p} \xi_j \geq 1, \xi \geq 0\}$. Then the local Kuhn–Tucker conditions imply that there exists a constant $\lambda(x) > 0$ satisfying the following system:

$$\left|\begin{array}{l} f_j(x) - \lambda(x) \prod_{l=1}^{p} \xi_l(x)/\xi_j(x) = 0\,, \quad j = 1, \ldots, p\,, \\ \prod_{j=1}^{p} \xi_j(x) = 1\,. \end{array}\right. \tag{2.9}$$

(Note that each $\xi_j(x)$ cannot be zero.) It follows from (2.9) that

$$\sum_{j=1}^{p} f_j(x)\xi_j(x) = p\sqrt[p]{\prod_{j=1}^{p} f_j(x)}\,. \tag{2.10}$$

Hence, solving (2.6) amounts to solve the original problem (P). Both (2.7) and (2.8) immediately follow from (2.9) and (2.10). □

Let us denote

$$\bar{\xi}_j = \min\{\sqrt[p]{g_0(x^l)} \mid l = 1, \ldots, p\}/f_j(x^j)\,, \quad j = 1, \ldots, p\,, \tag{2.11}$$

where x^j's represent optimal solutions of the respective problems (2.4).

COROLLARY 2.2.

$$1\Big/\prod_{l \neq j} \bar{\xi}_l \leq \xi_j^* \leq \bar{\xi}_j\,, \quad j = 1, \ldots, p\,. \tag{2.12}$$

Proof. The second inequality is derived from (2.8) and the following relations:

$$g_0(x^*) \leq g_0(x^j)\,, \quad j = 1, \ldots, p\,,$$

$$f_j(x^*) \geq f_j(x^j)\,, \quad j = 1, \ldots, p\,.$$

The first follows from

$$\xi_j^* \prod_{l \neq j} \bar{\xi}_l \geq \prod_{l=1}^{p} \xi_l^* \geq 1\,, \quad j = 1, \ldots, p\,. \qquad \square$$

Let us consider a subproblem of (2.6):

$$\mathrm{P}(\xi) \left| \begin{array}{ll} \text{minimize} & G(x;\xi) = \sum_{j=1}^{p} \xi_j f_j(x) \\ \text{subject to} & x \in X\,, \end{array} \right.$$

where $\xi > 0$ is a constant vector. $\mathrm{P}(\xi)$ is a convex minimization problem for any $\xi > 0$. Let $x^*(\xi)$ be an optimal solution of $\mathrm{P}(\xi)$ and let

$$h(\xi) \equiv G(x^*(\xi);\xi)\,. \tag{2.13}$$

Then (2.6) can be reduced to the following problem with p variables:

$$(\mathrm{MP}) \left| \begin{array}{ll} \text{minimize} & h(\xi) \\ \text{subject to} & \prod_{j=1}^{p} \xi_j \geq 1\,, \quad \xi \geq 0\,. \end{array} \right. \tag{2.14}$$

THEOREM 2.3. *h is a concave function over $\xi > 0$ and has the following properties*:

$$h(\lambda\xi) = \lambda h(\xi)\,, \quad \forall \lambda \geq 0\,, \tag{2.15}$$

$$h(\xi^1) \leq h(\xi^2) \quad \textit{if} \quad 0 < \xi^1 \leq \xi^2\,. \tag{2.16}$$

Proof. Choose arbitrary $\xi^1, \xi^2 > 0$ and let

$$x^k = \operatorname{argmin}\{G(x;\xi^k) \mid x \in X\}\,, \quad k = 1, 2\,,$$

$$x^3 = \operatorname{argmin}\{G(x;\lambda\xi^1 + (1-\lambda)\xi^2) \mid x \in X\}\,,$$

where $\lambda \in [0, 1]$. Then we have

$$\begin{aligned} h(\lambda\xi^1 + (1-\lambda)\xi^2) &= \sum_{j=1}^{p} (\lambda\xi_j^1 + (1-\lambda)\xi_j^2) f_j(x^3) \\ &= \lambda \sum_{j=1}^{p} \xi_j^1 f_j(x^3) + (1-\lambda) \sum_{j=1}^{p} \xi_j^2 f_j(x^3) \\ &\geq \lambda \sum_{j=1}^{p} \xi_j^1 f_j(x^1) + (1-\lambda) \sum_{j=1}^{p} \xi_j^2 f_j(x^2) \\ &= \lambda h(\xi^1) + (1-\lambda) h(\xi^2)\,. \end{aligned}$$

(2.15) and (2.16) are obvious. $\square$

Let us denote by Ξ the feasible region of (MP), i.e.,

$$\Xi = \left\{ \xi \in R^p \;\middle|\; \prod_{j=1}^{p} \xi_j \geqslant 1,\ \xi \geqslant 0 \right\}. \tag{2.17}$$

COROLLARY 2.4. *There exists a globally optimal solution of* (MP) *among the boundary points of* Ξ.

Proof. Obvious from Theorem 2.3.

3. Outer Approximation Algorithm for the Master Problem

Let us proceed to the algorithm for solving the master problem (MP). By Corollary 2.2, a globally optimal solution ξ^* of (MP) is contained in a p-dimensional cube:

$$\Xi_0 = \{ \xi \in R^p \mid \underline{\xi} \leqslant \xi \leqslant \bar{\xi} \}, \tag{3.1}$$

where

$$\begin{cases} \underline{\xi} = (1/\Pi_{l \neq 1} \bar{\xi}_l, \ldots, 1/\Pi_{l \neq p} \bar{\xi}_l)^t, \\ \bar{\xi} = (\bar{\xi}_1, \ldots, \bar{\xi}_p)^t \end{cases} \tag{3.2}$$

and $\bar{\xi}_j$'s are defined by (2.11). Thus, (MP) is equivalent to the following:

$$\left| \begin{array}{ll} \text{minimize} & h(\xi) \\ \text{subject to} & \xi \in \Xi \cap \Xi_0 . \end{array} \right. \tag{3.3}$$

Since $\Xi \cap \Xi_0$ is a nonempty, convex and compact set, it is well known [6] that ξ^* can be obtained by applying an outer approximation method. Fortunately, we can solve (MP) more efficiently by using its special structures stated in the previous section.

Let

$$(\mathrm{P}_0) \left| \begin{array}{ll} \text{minimize} & h(\xi) \\ \text{subject to} & \xi \in \Xi_0 \end{array} \right. \tag{3.4}$$

be the initial relaxed problem. By (2.16) of Theorem 2.3, a globally optimal solution ξ^0 of (P_0) is immediately obtained, i.e.,

$$\xi_j^0 = 1 \Big/ \prod_{l \neq j} \bar{\xi}_l , \quad j = 1, \ldots, p . \tag{3.5}$$

Assume that we obtain the following kth relaxed problem:

$$(\mathrm{P}_k) \left| \begin{array}{ll} \text{minimize} & h(\xi) \\ \text{subject to} & \xi \in \Xi_k , \end{array} \right. \quad k = 1, 2, \ldots \tag{3.6}$$

where

$$\Xi_0 \supset \Xi_k \supset \Xi \cap \Xi_0 . \tag{3.7}$$

Let ξ^k be an optimal solution of (P_k).

LEMMA 3.1.

$$h(\xi^k) \leq h(\xi^*) \leq h(\xi^k) \Big/ \sqrt[p]{\prod_{j=1}^{p} \xi_j^k} \,. \tag{3.8}$$

Proof. Since $\xi^k / \sqrt[p]{\prod_{j=1}^{p} \xi_j^k} \in \Xi$, we have

$$h(\xi^*) \leq h\left(\xi^k \Big/ \sqrt[p]{\prod_{j=1}^{p} \xi_j^k}\right).$$

The second inequality of (3.8) is derived from this and (2.15) of Theorem 2.3. The first inequality is obvious. □

For each ξ^k let us define a function:

$$l_k(\xi) = \sqrt[p]{\sum_{j=1}^{p} \xi_j^k} \sum_{j=1}^{p} \xi_j / \xi_j^k - p \,. \tag{3.9}$$

Note that $l_k(\xi) = 0$ is a supporting hyperplane of Ξ at $\xi^k / \sqrt[p]{\prod_{j=1}^{p} \xi_j^k}$ and that

$$l_k(\xi^k) < 0 \,, \quad l_k(\xi) \geq 0 \,, \quad \forall \xi \in \Xi \tag{3.10}$$

if $\xi^k \not\in \Xi$. By using l_k, the kth cut L_k can be defined as follows:

$$L_k = \{ \xi \in R^p \mid l_k(\xi) \geq 0 \} \,. \tag{3.11}$$

Figure 1 shows the relation between L_k's and the feasible region Ξ of (MP).

We are now ready to construct the outer approximation method for (MP) with a given tolerance $0 \leq \epsilon < 1$:

ALGORITHM OAM

Step 0. Let $k = 0$.

Step 1. Compute an optimal solution ξ^k of a relaxed problem (P_k).

Step 2. If

$$\prod_{j=1}^{p} \xi_j^k + \epsilon \geq 1 \,,$$

then stop. Otherwise, generate a cut L_k from (3.9) and (3.11) and let

$$\Xi_{k+1} = \Xi_k \cap L_k \,, \quad k = k + 1 \,.$$

Return to Step 1. □

THEOREM 3.2. *If $\epsilon > 0$, Algorithm OAM terminates after finitely many iterations and yields an approximate solution*:

$$\xi^\epsilon = \xi^k \Big/ \sqrt[p]{\prod_{j=1}^{p} \xi_j^k} \,, \tag{3.12}$$

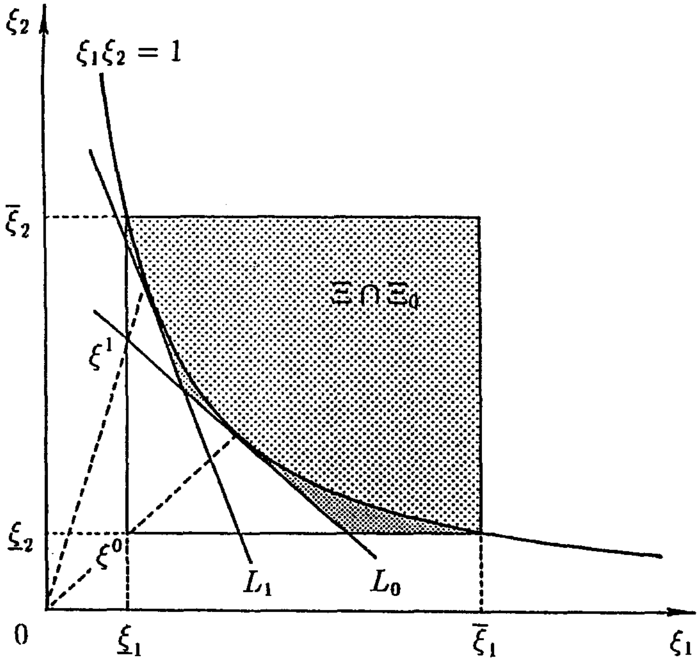

Fig. 1. The relation between the cuts and the feasible region.

which satisfies

$$\sqrt[p]{1-\epsilon}\,h(\xi^\epsilon) \leq h(\xi^*) \leq h(\xi^\epsilon)\,. \tag{3.13}$$

If $\epsilon = 0$, OAM *generates a sequence* ξ^k, $k = 0, 1, 2, \ldots$, *every accumulation point of which is a globally optimal solution of* (MP).

Proof. By Lemma 3.1, we have

$$\sqrt[p]{\prod_{j=1}^{p} \xi_j^k}\,h(\xi^\epsilon) \leq h(\xi^*) \leq h(\xi^\epsilon)\,.$$

If OAM terminates, then $\Pi_{j=1}^{p} \xi_j^k \geq 1-\epsilon$ must hold. Therefore, ξ^ϵ satisfies (3.13).

Now assume that OAM is infinite. Then there exists a positive constant θ and a subsequence ξ^{k_q}, $k_q \in \{0, 1, 2, \ldots\}$ such that

$$1 - \prod_{j=1}^{p} \xi_j^{k_q} > \theta\,, \quad \forall q\,. \tag{3.14}$$

Since all ξ_k's are contained in the compact set Ξ_0, we may assume that the sequence ξ^{k_q}, $k_q \in \{0, 1, 2, \ldots\}$ converges to $\tilde{\xi}$. Let

$$l(\xi) = \sqrt[p]{\prod_{j=1}^{p} \tilde{\xi}_j}\,\sum_{j=1}^{p} \frac{\xi_j}{\tilde{\xi}_j} - p\,. \tag{3.15}$$

Then

$$\lim_{q\to\infty} l_{k_q}(\xi^{k_{q+1}}) = \lim_{q\to\infty} l_{k_q}(\xi^{k_q}) = l(\tilde{\xi}) \geqslant 0$$

because $\xi_{k_{q+1}} \in \Xi_{k_{q+1}} \subset \Xi_{k_q} \cap L_{k_q}$ for any q. Hence, by (3.15)

$$l(\tilde{\xi}) = p\left(\sqrt[p]{\prod_{j=1}^{p} \tilde{\xi}_j} - 1\right) \geqslant 0 ,$$

which contradicts (3.14). If $\epsilon > 0$, therefore, OAM must terminate after finitely many interactions. If $\epsilon = 0$, then

$$h(\tilde{\xi}) = \lim_{q\to\infty} h(\xi^{k_q}) \leqslant h(\xi^*)$$

because $h(\xi^{k_q}) \leqslant h(\xi^*)$ for any q. □

Denote by $V(\Xi_k)$ the vertex set of Ξ_k. Since (P_k) is a concave minimization problem (Theorem 2.3), there exists a globally optimal solution ξ^k of (P_k) among $V(\Xi_k)$, i.e.,

$$\xi^k = \operatorname{argmin}\{h(\xi) \mid \xi \in V(\Xi_k)\} . \tag{3.16}$$

Therefore, ξ^k can be computed by solving the convex program $\mathrm{P}(\xi)$ for every $\xi \in V(\Xi_k)$. For each $k = 1, 2, \ldots$, we can compute $V(\Xi_k)$ by $V(\Xi_{k-1})$ and L_{k-1}. Let V_k be the vertex set which is generated by adding the cut L_{k-1} to Ξ_{k-1}. Then

$$V(\Xi_k) = V_k \cup \{\xi \in V(\Xi_{k-1}) \mid \xi \in L_{k-1}\} . \tag{3.17}$$

Several methods are available for finding V_k. Refer to [6] for the details.

4. Computational Experiments

We report the results of computational experiments of Algorithm OAM presented in the previous section. We solved examples belonging to either one of the following two subclasses of (P):

$$\left|\begin{array}{ll} \text{minimize} & \prod_{j=1}^{p} c_j^t x \\ \text{subject to} & Ax \geqslant b , \quad x \geqslant 0 , \end{array}\right. \tag{4.1}$$

$$\left|\begin{array}{ll} \text{minimize} & c_0^t(x) \cdot \prod_{j=1}^{q} [c_j^t x + x^t \operatorname{diag}(d_j) x] \\ \text{subject to} & Ax \geqslant b , \quad x \geqslant 0 , \end{array}\right. \tag{4.2}$$

where $A \in R^{m\times n}$, $b \in R^m$, $c_j \in R^n$ and $d_j \in R^n$. All data were randomly generated, whose ranges are [0, 100]. We employed the method proposed in [21] to find new vertices V_k generated by adding the cut L_{k-1} to Ξ_{k-1}. For each $\xi \in V_k$ the subproblems $\mathrm{P}(\xi)$'s of (4.1) and (4.2) are a linear program and a convex quadratic program, respectively. We applied the revised simplex method to the former while the latter was solved by the reduced gradient method. The tolerance was always fixed at $\epsilon = 10^{-5}$ and ten examples were solved for each size of the problems. The programs were coded in C language and tested on a SUN4/75 workstation.

Tables I and II show the results for (4.1) when the size of p was fixed at 3 and 4, respectively. The average CPU time (and its standard deviation) required by OAM is listed. They also contain the average number of cuts and vertices generated in the course of computation. The number of vertices corresponds to that of subproblems solved for each example. Table III shows the results for (4.1) when $(m, n) = (20, 30)$ and the size of p ranges from 2 to 5. The results for (4.2) when $q = 2$ and $q = 3$ are listed in Tables IV and V, respectively.

We see from these tables that Algorithm OAM is very sensitive to the size of p. The number of cuts and vertices generated throughout computation sharply increases as a function of p. This is partly due to the inefficiency of the implementation for finding vertices V_k. However, it should be noted that these figures increase slowly as the size of (m, n) gets larger. It is also worth noting that the number of cuts for (4.1) and (4.2) are quite similar (when $p = 4$ and $q = 3$,

Table 1. Results for (4.1) when $p = 3$

m	80	100	100	120	120	150	150	200
n	100	100	120	120	140	140	160	180
Average CPU time in seconds (standard deviation)								
	43.98	59.12	115.25	178.57	181.43	381.21	427.02	914.09
	(9.05)	(17.80)	(28.02)	(43.13)	(41.12)	(93.80)	(127.63)	(129.88)
Av. # of cuts (s.d.)								
	41.8	38.3	46.9	47.5	45.5	46.1	46.1	42.5
	(5.98)	(4.67)	(12.51)	(7.47)	(7.23)	(9.98)	(9.67)	(3.75)
Av. # of vertices (s.d.)								
	176.5	159.0	200.0	204.9	197.1	200.4	200.0	180.3
	(30.91)	(23.14)	(61.42)	(37.07)	(36.99)	(46.86)	(50.55)	(19.29)

Table II. Results for (4.1) when $p = 4$

m	50	50	60	80	100	100	120
n	40	60	80	100	100	120	120
Av. CPU time in seconds (s.d.)							
	49.05	95.05	155.10	330.55	524.49	617.51	1154.83
	(46.44)	(32.49)	(66.54)	(101.87)	(210.27)	(141.65)	(381.51)
Av. # of cuts (s.d.)							
	77.9	81.9	86.8	100.1	101.5	98.5	99.8
	(21.60)	(11.41)	(15.09)	(17.84)	(24.62)	(13.68)	(18.65)
Av. # of vertices (s.d.)							
	983.7	1060.6	1153.8	1386.0	1414.7	1370.6	1385.3
	(365.13)	(199.37)	(258.10)	(311.34)	(422.30)	(251.71)	(327.60)

Table III. Results for (4.1) when $(m, n) = (20, 30)$

p	2	3	4	5
Av. CPU time in seconds (s.d.)				
	0.46	1.27	14.21	1170.36
	(0.05)	(0.25)	(10.46)	(950.53)
Av. # of cuts (s.d.)				
	10.1	37.4	62.8	118.3
	(3.78)	(4.20)	(12.66)	(19.30)
Av. # of vertices (s.d.)				
	19.2	152.5	733.2	5406.2
	(7.56)	(20.42)	(207.12)	(1585.29)

Table IV. Results for (4.2) when $q = 2$

m	50	50	60	80	100	100	120
n	40	60	80	100	100	120	120
Av. CPU time in seconds (s.d.)							
	25.12	100.61	239.44	659.80	685.04	1268.57	1801.33
	(25.44)	(71.23)	(88.88)	(532.53)	(303.05)	(680.56)	(1136.87)
Av. # of cuts (s.d.)							
	34.6	45.5	43.1	43.7	43.0	52.7	51.4
	(8.62)	(19.41)	(12.51)	(10.63)	(14.72)	(10.74)	(17.60)
Av. # of vertices (s.d.)							
	140.7	192.9	181.9	185.3	181.3	226.9	222.6
	(40.71)	(99.68)	(63.14)	(50.54)	(70.93)	(51.36)	(87.26)

Table V. Results for (4.2) when $q = 3$

m	30	30	50	50	60
n	20	40	40	60	80
Av. CPU time in seconds (s.d.)					
	53.29	230.57	716.78	1667.11	3089.74
	(21.43)	(152.24)	(491.19)	(971.76)	(1485.14)
Av. # of cuts (s.d.)					
	69.3	93.6	91.1	109.1	101.3
	(12.84)	(27.52)	(13.63)	(26.09)	(23.39)
Av. # of vertices (s.d.)					
	829.1	1234.2	1194.6	1505.6	1352.6
	(199.55)	(473.77)	(236.40)	(439.97)	(378.39)

respectively.) This implies that the total computational time is dominated by that needed for solving the convex program, i.e., the subproblem $P(\xi)$.

We concluded from this that our algorithm is reasonably efficient when p is less than 4. When p is over 5, we need more efficient procedures for finding V_k [3] and for solving the associated convex program $P(\xi)$.

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